332:541 **Stochastic Signals and Systems** **Fall 2000**

Quiz I and Solutions

You have 1.5 hours to answer the following three questions. You are allowed both sides of an 11x8.5 inch piece of paper for notes. GOOD LUCK!

1. *(50 points)*

   Derived Random Variables: Let X and Y be two independent uniformly distributed random variables on the interval $[0,1]$. Let $Z = X^2 + Y^2$. Please find the CDF and thence the probability density for the random variable Z. Sketch the density function.

   CDF for $A = X^2$ is obtained from

$$F_A(a) = \int_0^{\sqrt{a}} dx$$

   which implies

$$f_A(a) = \frac{1}{2}a^{-1/2}$$

   on $(0,1)$ and zero elsewhere. Since X and Y are independent the density on Z is the convolution of the densities for X^2 and Y^2 (which we just derived). There are two intervals to consider. On $0 \le z \le 1$ we have

$$f_Z(z) = \frac{1}{4}\int_0^z (z-a)^{-1/2}(a)^{-1/2}da =$$

   and on $1 \le z \le 2$ we have

$$f_Z(z) = \frac{1}{4}\int_{z-1}^1 (z-a)^{-1/2}(a)^{-1/2}da =$$

   and that is as far as you can go using this approach unless you have an excellent command of integral calculus (which I don't), an integral book or a symbolic math program. If you went this way and formulated it this far, you got almost full credit. If you went on to get it completely correct you got full credit and a smile from me for being a calculus wizard.

   In any case, according to MAPLE©, the integral of $\int \frac{1}{\sqrt{x(c-x)}}dx$ is

$$\arctan(\frac{x - c/2}{\sqrt{xc - x^2}})$$

   so that

$$f_Z(z) = \pi/4$$

   on $[0,1]$ (which is EXTREMELY counterintuitive until you look at the next solution). For $[1,2]$ we have

$$f_Z(z) = \frac{1}{4}\left[\arctan(\frac{1 - z/2}{\sqrt{z-1}}) - \arctan(\frac{(z-1) - z/2}{\sqrt{(z-1)z - (z-1)^2}})\right]$$

   Another approach is to look at the JOINT probability space (which is a little square in the (x,y)) plane and calculating the CFD of Z directly by integrating the appropriate region in the (x,y) plane.

For $0 \leq z \leq 1$ the region is a 1/4-pie (like the circular pie you eat) slice of radius $\sqrt{z}$. For a slice which subtends angle θ radians with radius r, the area is $\frac{\theta}{2}r^2$ This yields

$$Prob(Z \leq z) = \frac{\pi}{4}z$$

which gives the PDF as $\pi/4$ on $(0,1)$.

For $1 \leq z \leq 2$ the pie slice exceeds the boundaries of the box. So you add up the areas of the two triangles (sides 1 and hypotenuse $\sqrt{z}$ for a total area of $\sqrt{z-1}$) and then the joining pie sliver of radius $\sqrt{z}$ which subtends an angle $(2)(\pi/4 - \arctan\sqrt{z-1})$ and therefore has area $z(\pi/4 - \arctan\sqrt{z-1})$. The CDF is then

$$Prob(Z \leq z) = \sqrt{z-1} + z(\pi/4 - \arctan\sqrt{z-1})$$

Differentiating in z gives

$$f_Z(z) = \pi/4 - \arctan(\sqrt{z-1})$$

which (unbelievably enough) is the same result as the first method.

Note that even if you did not know the derivative of arctan, this approach allowed you to sketch the PDF since you SHOULD know how arctan() *behaves more or less even if you don't know the integral.*

2. *(50 points)* **A Blast From the Recent Past:** Suppose an experiment is performed N times independently and that the probability of event $\mathcal{A}$ occuring is p. We then know that if k is the number of times event $\mathcal{A}$ occurs in N trials that by the (strong) law of large numbers we have

$$\lim_{N\to\infty} \frac{k}{N} = p$$

   (a) *(25 points)* Let X be a discrete random variable which can take on a finite number of different values a_m, $m = 1, 2, ..., M$ each with probability p_m, $\sum_m p_m = 1$. Suppose we have N independently observed values of X, $x_1, x_2, ...x_N$. Use the above law to show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i = E(X)$$

   Let k_m be the number of times event m occurred in the N trials. Then $\lim_{N\to\infty} k_m/N = p_m$. Thus we can form

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i = \lim_{N\to\infty} \sum_{m=1}^{M} a_m \frac{k_m}{N} = \sum_{m=1}^{M} a_m p_m = E(X)$$

   (b) *(25 points)* If possible, repeat part (a) for a continuous random variable, but one which is limited to taking on values on a finite interval, say (0,1).

   Let the continuous random variable be X with probability density $f_X(x)$. Form bins of size $1/M$ and essentially quantize the sample values x_i to values $a_m = m/M$, $m = 1, 2, \cdots, M$. Call the derived (discrete) random variable $\tilde{X}_i$. We then define $N = QM$ for an integer Q and the difference $V_i = X_i - \tilde{X}_i$. We then have

$$p_m = Prob((m-1)/M \leq \tilde{X}_i \leq m/M) = \int_{(m-1)/M}^{m/M} f_X(x)dx$$

and from the first part we know that

$$\lim_{Q\to\infty} \frac{1}{QM} \sum_{i=1}^{QM} \tilde{x}_i = \sum_{m=1}^{M} a_m p_m = E(\tilde{X})$$

We then write

$$\lim_{Q\to\infty} \frac{1}{QM} \sum_{i=1}^{QM} \tilde{x}_i = \lim_{Q\to\infty} \frac{1}{QM} \sum_{i=1}^{QM} (X_i - V_i) = \lim_{Q\to\infty} \frac{1}{QM} \sum_{i=1}^{QM} X_i - \lim_{Q\to\infty} \frac{1}{QM} \sum_{i=1}^{QM} V_i$$

and note that we must have $|V_i| \leq 1/M$. *This means that if we artfully construct* $Q = \sqrt{M}$ *we have*

$$\left| \lim_{M\to\infty} \frac{1}{M^{3/2}} \sum_{i=1}^{M^{3/2}} V_i \right| \leq \lim_{M\to\infty} 1/M = 0$$

Likewise we note that from the definition of p_m *and* a_m *we have*

$$\lim_{M\to\infty} \sum_{m=1}^{M} a_m p_m = E[X]$$

and we're done.

3. *(50 points)*

Rutgera and Wall Street: Rutgera Universa, the world famous Rutgers University Graduate student has a part time job on Wall Street with an investment bank. Rutgera's job is to find promising investment strategies and make money for her bosses.

Rutgera is approached one day with the followed scheme.

- Rutgera provides capital (money) C
- At the end of the day Rutgera's portfolio will have value $P = \alpha C$ where Prob$(\alpha = 2) = 1/2$, Prob$(\alpha = 1/2) = 1/4$ and Prob$(\alpha = 1/4) = 1/4$.
- The same rules apply on any given day and she can reinvest whatever the total was at the end of the previous day (i.e., $C(n) = P(n-1)$). Each day's events are independent of all ohers

You must help Rutgera decide whether this is a good or bad investment scheme.

(a) *(10 points)* Assuming Rutgera puts $C(1) = Q$ dollars in at the beginning of day 1, carefully sketch the probability mass function for $P(1)$ and calculate its expected value. Calculate its variance.

Prob$(P(1) = 2Q) = 1/2$, *Prob*$(P(1) = Q/2) = 1/4$ *and Prob*$(P(1) = Q/4) = 1/4$. *The expected value is* $Q(2(1/2) + (1/2)(1/4) + (1/4)(1/4)) = (19/16)Q$. *The variance is* $Q^2[(13/16)^2(1/2) + (11/16)^2(1/4) + (15/16)^2(1/4)] = 684Q^2/1024 = 0.668Q^2$

(b) *(10 points)* Each day Rutgera reinvests the whole portfolio $C(n) = P(n-1)$. Let α_n be the scale factor for day n. Find an expression for $P(n)$ in terms of $C(1)$ and the α_i. What is the expected value and variance of $P(n)$ assuming $C(1) = Q$.

$P(n) = Q^n \prod_{i=1}^{n} \alpha_i$. *Since the* α_i *are independent* $E(P(n)) = Q[E(\alpha)]^n = Q(19/16)^n$. *The variance is*

$$Q^{2n} E[(\prod_{i=1}^{n} \alpha_i - (19/16)^n)^2] = Q^{2n} \left[\prod_{i=1}^{n} E[\alpha_i^2] - 2(19/16)^2 \prod_{i=1}^{n} E[\alpha_i] + (19/16)^{2n} \right]$$

which simplifies to

$$\sigma^2_{P(n)} = Q^{2n}\left[(133/64)^n - (19/16)^{2n}\right] = Q^2\left[(133/64)^n - (19/16)^{2n}\right]$$

(c) *(10 points)* Define $L(n) = \log P(n)$. Calculate the mean and variance of $L(n)$.
$P(n) = Q^n \prod_{i=1}^n \alpha_i$. *Since the* α_i *are independent* $E[\log(P(n))] = \log(Q) + nE\log(\alpha) = n\log(Q) - n/4$. *The variance is the sum of variances of the individual* α_i. *Since they are i.i.d we have*

$$\sigma^2_{\log(P(n))} = n\sigma^2_\alpha = n[(5/4)^2/2 + (-3/4)^2/4 + (-7/4)^2/4] = (108/64)n = (27/16)n$$

(d) *(20 points)* Is this a good or bad investment scheme? HINT: This is an application of the law of large numbers.

It looks like a good investment scheme at first because the expected value of the portfolio on day n *is* $Q^n\beta^n$ *where* $\beta = 19/16 > 1$. *That is, the expected value of the portfolio goes to infinity in* n. *However, the fact that the variances also go to infinity is cause for concern.*

So we note that the log portfolio is a sum of independent identically distributed random variables. That means that the sample mean goes to the expected value WITH PROBABILITY 1!!!!! The expected value is -1/4 which means that $\log(P(n))/n \to -1/4$. *Thus, all the money is lost with probability 1. Ergo, the investment scheme SUCKS!SUCKS!SUCKS!*

Note: this problem was slightly different from previous versions on previous exams in that you did not need the central limit theorem (because you don't know that theorem yet). If you used the CLT, you did not get credit.