332:541 **Stochastic Signals and Systems** **Fall 2000**

More Homework Solutions

Here are solutions to the three homework problems over two assignments. The first two are hypothesis testing problems. The last is estimation of coin toss probability.

1. You are given three hypotheses H_i, $i = 0, 1, 2$ with *a priori* probabilities p_i, $\sum_i p_i = 1$. Find a decision rule which minimizes the probability of error.

   Well, first we have to form the cost metric for probability of error. This is

$$\mathcal{C} = \sum_i p_i \left(1 - \int_{H_i} f_{X|H_i}(x|H_i)dx\right) \tag{1}$$

   We then note that $\cup_i H_i = \Re$ so that we may rewrite this expression as

$$\mathcal{C} = p_0\left(1 - \int_{H_0} f_{X|H_0}(x|H_0)dx\right) + p_1\left(1 - \int_{H_1} f_{X|H_1}(x|H_1)dx\right) + p_2\int_{H_0} f_{X|H_2}(x|H_2)dx + p_2\int_{H_1} f_{X|H_2}(x|H_2)dx \tag{2}$$

   Recombining we have

$$\mathcal{C} = p_0 + p_1 + \int_{H_0}\left(p_2 f_{X|H_2}(x|H_2) - p_0 f_{X|H_0}(x|H_0)\right)dx + \int_{H_1}\left(p_2 f_{X|H_2}(x|H_2) - p_1 f_{X|H_1}(x|H_1)\right)dx \tag{3}$$

   which allows us to see that we'll only choose $x \in H_0$ when $p_2 f_{X|H_2}(x|H_2) - p_0 f_{X|H_0}(x|H_0) < 0$ which translates into

$$\frac{p_0 f_{X|H_0}(x|H_0)}{p_2 f_{X|H_2}(x|H_2)} \underset{\text{say } H_2 \text{ over } H_0}{\overset{\text{say } H_0 \text{ over } H_2}{\gtrless}} 1 \tag{4}$$

   and similarly

$$\frac{p_1 f_{X|H_1}(x|H_1)}{p_2 f_{X|H_2}(x|H_2)} \underset{\text{say } H_2 \text{ over } H_1}{\overset{\text{say } H_1 \text{ over } H_2}{\gtrless}} 1 \tag{5}$$

   Finally we realize we could have reworked the above equations starting with H_1 and we obtain

$$\frac{p_1 f_{X|H_1}(x|H_1)}{p_0 f_{X|H_0}(x|H_0)} \underset{\text{say } H_0 \text{ over } H_1}{\overset{\text{say } H_1 \text{ over } H_0}{\gtrless}} 1 \tag{6}$$

   These three relations define the optimal decision regions for minimum probability of error.

2. The coin dropping problem had two probability distributions. Given H_A we have X distributed as a 2-d gaussian with zero mean and variance σ_A^2 and for H_B, X is distributed as a 2-d gaussian with zero mean and variance σ_B^2. I seem to remember saying the *a priori* probabilities were equal, but I'll solve the problem in general with $p_A + p_B = 1$.

   We form our likelihood ratio

$$\frac{\frac{p_A}{2\pi\sigma_A^2} e^{-(x^2+y^2)/2\sigma_A^2}}{\frac{p_B}{2\pi\sigma_B^2} e^{-(x^2+y^2)/2\sigma_B^2}} \underset{\text{say } H_B}{\overset{\text{say } H_A}{\gtrless}} 1 \tag{7}$$

Moving things around and taking the log we get

$$\frac{r^2}{2\sigma_B^2} - \frac{r^2}{2\sigma_A^2} \begin{array}{c} \text{say } H_A \\ > \\ < \\ \text{say } H_B \end{array} \log\left(\frac{p_B\sigma_A^2}{p_A\sigma_B^2}\right) \tag{8}$$

where $r^2 = x^2 + y^2$. *Thus the decision region will be a disc centered at the origin and its complement.*

3. Finally we have our estimation problem. You were asked to estimate the probabiliyh of a coin turning up heads given N tosses worth of observation. But specifically, you were asked for the LINEAR estimate. You were given $f_P(p)$ was uniform on $(0, 1)$

   Well, what we need in this case is an estimate for p which is a scaled superposition of all the observations x_i. *So, first we set* $x_i = 0$ *if toss i yields tails and* $x_i = 1$ *if not. Prob(*$x_i = 1$*)* $= p$ *(obviously). Maybe not as obvious, we could have assigned ANY different finite values to these variables. But as you follow through I think you'll see this is so (and why we chose this particular assignment).*

   Our estimate is then

$$\hat{p} = \sum_{i=1}^{N} a_i x_i + b \tag{9}$$

   and we're left to find the proper a_i *and* b. *We first form the error metric and the (blindly) differentiate and set to zero. I say blindly since I've tried to drum into you that it's dangerous to assume convexity in multivariate functions. With that warning, we then have*

$$e^2 = E\left[\left(\sum_{i=1}^{N} a_i x_i + b - p\right)^2\right] \tag{10}$$

   Differentiating we have

$$\frac{\partial e^2}{\partial a_j} = E\left[2x_j\left(\sum_{i=1}^{N} a_i x_i + b - p\right)\right] \tag{11}$$

   and

$$\frac{\partial e^2}{\partial b} = E\left[2\left(\sum_{i=1}^{N} a_i x_i + b - p\right)\right] \tag{12}$$

   We see for $i \neq j$ *that* $E[X_i X_j | P = p] = p^2$ *and for* $i = j$ *we have* $E[X_i | P = p] = E[X_i^2 | P = p] = p$. *Taking the expectation over* P *yields* $E[X_i X_j] = 1/3$ *and* $E[X_i^2] = 1/2$. *We also see that*

$$E[X_i p] = \int_{x=0}^{1}\int_{0}^{1} xp[(1-p)\delta(x) + p\delta(x-1)]dpdx = \int_{0}^{1} p^2 dp = \frac{1}{3} \tag{13}$$

   So all told we have

$$\frac{\partial e^2}{\partial a_j} = \frac{1}{3}\sum_{i\neq j}^{N} a_i + \frac{1}{2}a_j + \frac{b}{2} - \frac{1}{3} = 0 \tag{14}$$

   and

$$\frac{\partial e^2}{\partial b} = \frac{1}{2}\sum_{i=1}^{N} a_i + b - \frac{1}{2} = 0 \tag{15}$$

   so that

$$b = \frac{1}{2}(1 - \sum_{i=1}^{N} a_i) \tag{16}$$

   so that

$$\frac{1}{3}\sum_{i\neq j}^{N} a_i + \frac{1}{2}a_j + \frac{1}{4}(1 - \sum_{i=1}^{N} a_i) - \frac{1}{3} = 0 \tag{17}$$

Simplifying

$$\frac{1}{12}\left(\sum_{i\neq j}^{N} a_i + 3a_j - 1\right) \tag{18}$$

Since this equation is identical for all a_j *it implies that the* a_j *are all identical. Therefore*

$$a_j = \frac{1}{N+2} \tag{19}$$

and

$$b = \frac{1}{N+2} \tag{20}$$