# 332:541 Solutions To Homework Problems (assigned on October 10, 2000)

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**Problem 1:** A certain radio receiver gets inputs $r(t) = \sum_{i=1}^{N} A_i \cos(\omega_c(t-\tau_i))$ where the $A_i$ are independent, identically distributed normal (Gaussian) random variables with zero mean and variance $\sigma^2$ and the $\tau_i$ are uniform random variables on $(0, T)$ where $T$ is the period of the sinusoid $(2\pi/\omega_c)$. What is the PDF for the received power $P = \int_0^T r^2(t)dt$?

**Solution:** By performing the integration that defines $P$ we get

$$P = \int_0^T \left( \sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \cos[\omega_c(t-\tau_i)(t-\tau_j)] \right) dt \tag{1}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \int_0^T \cos[\omega_c(t-\tau_i)(t-\tau_j)]dt$$

The integral above results in

$$\int_0^T \cos[\omega_c(t-\tau_i)(t-\tau_j)]dt = \frac{T}{2} \cos[\omega_c(\tau_j - \tau_i) \tag{2}$$

which implies that $P$ is given by

$$P = \sum_{i=1}^{N} \sum_{j=1}^{N} A_i A_j \frac{T}{2} \cos[\omega_c(\tau_j - \tau_i)] \tag{3}$$

Assuming that the $\tau$'s are given, relation (3) implies that $P$ is a sum of products of random variables $A_i A_j$. Since $A_i$ and $A_j$ are independent with normal distribution (zero-mean and variance $\sigma^2$) we can compute the mean and variance of each of these products as

$$E[A_i A_j] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy \tag{4}$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy = 0$$

$$E[(A_i A_j)^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 y^2 \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} dx dy = \sigma^4 \tag{5}$$

When $\tau$'s are uniform random variables in $(0, T)$ and we take expectations for each cosine term in (3) with respect to them, all the cross terms for $i \neq j$ are equal to zero and vanish, remaining only those for $i = j$ for which the expected value is $1/T^2$. This implies that $P$ can be written as

$$P = \sum_{i=1}^{N} \frac{A_i^2}{2} \tag{6}$$

Note that each Gaussian in (6) has zero mean and variance $\sigma/2$ and after some algebra (see for example Appendix A.3 in [1]) the pdf for $P$ is onbtained as

$$f_P(p) = \frac{1}{(\sigma^2)^{N/2}\Gamma(N/2)} p^{N/2-1} e^{\frac{-p}{\sigma^2}}, \quad p \geq 0 \tag{7}$$

and the Gamma functions is

$$\Gamma(k) = \int_0^{\infty} u^{k-1} e^{-u} du = (k-1)! \tag{8}$$

if $k$ is a positive integer. Fon $N$ even we get a **central chi-square** pdf for $P$

$$f_P(p) = \frac{1}{(\sigma^2)^M (M-1)!} p^{M-1} e^{\frac{-p}{\sigma^2}}, \quad p \geq 0 \tag{9}$$

Of course, that in the limit for large $N$, the central limit theorem still applies, and a Gaussian distribution approximates the resulting pdf.

**Problem 2:** Suppose $Z = X^2 + Y^2$ where $X$ and $Y$ are zero mean normal variables with variance $\sigma^2$. What is the PDF of $Z$.

Compare your answer to what you obtained in the first problem. No they're not exactly the same, but I still want you to compare them.

**Solution:** This is a special case of finite sum of squares of normal random variables, with zero mean and variance $\sigma^2$, with $N = 2 \Rightarrow M = 1$. For such a case the central chi-square distribution particularizes to an exponential distribution

$$p_Z(z) = \frac{1}{2\sigma^2} e^{\frac{z}{2\sigma^2}} \tag{10}$$

Basically, the random variables $P$ (in first proble) and $Z$ (in the second one) have a similar type of distribution (central chi-square if $N$ is even in the first case). This should be expected since for the second problem variable $Z$ can also be interpreted as some sort of received power (for discrete channel; signal at the receiver is $x + y$ with $x, y$ normal r.v., then $X^2 + Y^2$ is also the power at the receiver).

## References


[1] G. Stuber, *Principles of Mobile Communication*, Kluwer Academic Publishers, Boston 1996.